Coulomb Gauge Quark Propagator from Lattice QCD

Mario Schröck

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Eberhard Karls Universität Tübingen

Work together with

- Giuseppe Burgio
- Markus Quandt
- Hugo Reinhardt

Outline

Continuum vs. lattice quark propagator

The free inverse continuum quark propagator, i.e. the Dirac operator, reads

$$\left(S_0^{-1}(p)\right)^{ab}_{\alpha\beta} = i(\gamma_\mu)_{\alpha\beta}\, p_\mu\, \delta^{ab} + m\, \delta_{\alpha\beta}\, \delta^{ab}.$$

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We use matrix/vector notation in Dirac space and omit color indices, where appropriate.

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Lattice

$$\mathcal{S}_0^{-1}(p) = i\gamma_\mu k_\mu + \left(m + \frac{1}{2}\hat{k}^2 a\right)\mathbb{1}.$$

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Interacting propagator

We are interested in how the structure of the free propagator in Coulomb gauge changes when the interactions with the gluon field are turned on

Inverse free quark propagator

$$S_0^{-1}(p) = i\gamma_4 p_4 \qquad + i\gamma_i p_i \qquad + \mathbb{1} m$$

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$$S^{-1}(p) = i\gamma_4 p_4 A_t(p) + i\gamma_i p_i A_s(p) + \mathbb{1} B_m(p) + i\gamma_4 p_4 \gamma_i p_i A_d(p).$$

Therefore we investigate four dimensionless dressing functions, to which we will refer to as temporal, spatial, massive and mixed component [Popovici, Watson, Reinhardt, 2008].

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$$A_t(p) = \frac{-i}{4N_c p_4^2} \operatorname{tr}\left[\gamma_4 p_4 S^{-1}(p)\right],$$

$$A_s(p) = \frac{-i}{4N_c \sum_i p_i^2} \operatorname{tr}\left[\gamma_i p_i S^{-1}(p)\right],$$

$$A_d(p) = \frac{-i}{4N_c p_4^2 \sum_i p_i^2} \operatorname{tr}\left[\gamma_4 p_4 \gamma_i p_i S^{-1}(p)\right],$$

$$B_m(p) = \frac{1}{4N_c} \operatorname{tr}\left[S^{-1}(p)\right].$$

Full quark propagator

$$S(p) = \frac{-i\gamma_4 p_4 A_t(p) - i\gamma_i p_i A_s(p) - i\gamma_4 p_4 \gamma_i p_i A_d(p) + B_m(p)\mathbb{1}}{D^2(p)},$$

where we defined

$$D^2(p) := p_4^2 A_t^2(p) + \sum_i p_i^2 A_s^2(p) + p_4^2 \sum_i p_i^2 A_d^2(p) + B_m^2(p).$$

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From above expression we can calculate, similar to the covariant case [Skullerud, Williams, 1999]

$$\mathcal{A}_t(p) := \frac{A_t(p)}{D^2(p)} = \frac{i}{4N_c p_4^2} \operatorname{tr}\left[\gamma_4 p_4 S(p)\right],$$

$$\mathcal{A}_s(p) := \frac{A_s(p)}{D^2(p)} = \frac{i}{4N_c \sum_i p_i^2} \operatorname{tr}\left[\gamma_i p_i S(p)\right],$$

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$$\mathcal{B}_m(p) := \frac{B_m(p)}{D^2(p)} = \frac{1}{4N_c} \operatorname{tr}\left[S(p)\right].$$

The propagator on the lattice

- On the lattice we calculate the bare propagator $\mathcal{S}_{\text{bare}}(a;p)$.

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- We expect the interacting lattice propagator to have a similar form to its continuum counterpart

$$\mathcal{S}^{-1}(p) = i\gamma_4 k_4 a A_t(p) + i\gamma_i k_i a A_s(p) + B_m(p)\mathbb{1} + i\gamma_4 k_4 \gamma_i k_i a^2 A_d(p).$$

Correlation functions of fermions on the lattice

The Euclidean lattice two-point function is given by

$$\left\langle \psi_\alpha^a(n)\overline{\psi}_\beta^b(m) \right\rangle = \frac{\int \mathcal{D}U\, \mathcal{D}\psi\, \mathcal{D}\overline{\psi}\; \psi_\alpha^a(n)\overline{\psi}_\beta^b(m)\, \mathrm{e}^{-S_F\left[\psi,\overline{\psi},U\right]-S_G[U]}}{\int \mathcal{D}U\, \mathcal{D}\psi\, \mathcal{D}\overline{\psi}\;\, \mathrm{e}^{-S_F\left[\psi,\overline{\psi},U\right]-S_G[U]}}$$

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use Wick's Theorem for Grassmann variables to yield

Lattice two-point function

$$\left\langle \psi^a_\alpha(n)\overline{\psi}^b_\beta(m)\right\rangle = \frac{\int \mathcal{D}U\;D^{-1}(n,m)^{ab}_{\alpha\beta}\,\det D\,\mathrm{e}^{-S_G[U]}}{a^4\int \mathcal{D}U\;\det D\,\mathrm{e}^{-S_G[U]}}$$

$D^{-1}(n,m)^{ab}_{\alpha\beta}$ is the inverse lattice Dirac operator.

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Lattice two-point function in quenched approximation $\det D = 1$

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- Reducing the problem: invert $D(n,m)$ for a given point-source $(\hat{n}\equiv 0,\hat{\alpha},\hat{a})$, i.e. solve

$$\sum_{m,\beta,b} D(n,m)^{ab}_{\alpha\beta}\mathcal{S}(m,0)^{b\hat{a}}_{\beta\hat{\alpha}} = \delta(n)\delta^{a\hat{a}}\delta_{\alpha\hat{\alpha}} \qquad \forall\,\hat{\alpha},\hat{a}.$$

to obtain the coordinate-space quark propagator.

Fourier Transformation

Fourier transform the Green's function $\mathcal{S}(n,0)$ to momentum-space

Momentum-space quark propagator

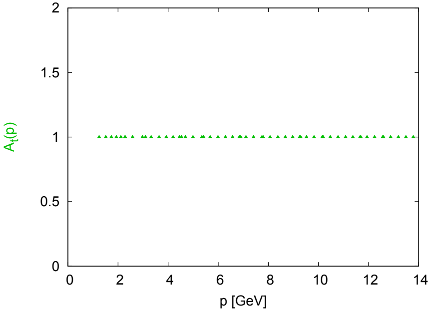
$$\mathcal{S}(p) = \sum_x \mathrm{e}^{-ip\cdot x}\mathcal{S}(x,0)\,,$$

where

$$x = na\,,$$
$$p_i = \frac{2\pi}{N_i a}\left(n_i + 1 - \frac{N_i}{2}\right)\,, \quad n_i = 0,\ldots,N_i,$$
$$p_4 = \frac{2\pi}{N_t a}\left(n_t + \frac{1}{2} - \frac{N_t}{2}\right)\,, \quad n_t = 0,\ldots,N_t.$$

How do the lattice dressing functions look like at tree-level?

$$\mathcal{S}^{-1}(p) = i\gamma_4 k_4 a A_t(p) + i\gamma_i k_i a A_s(p) + B_m(p)\mathbb{1} + i\gamma_4 k_4 \gamma_i k_i a^2 A_d(p).$$



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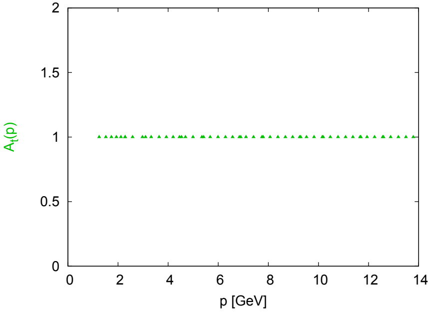

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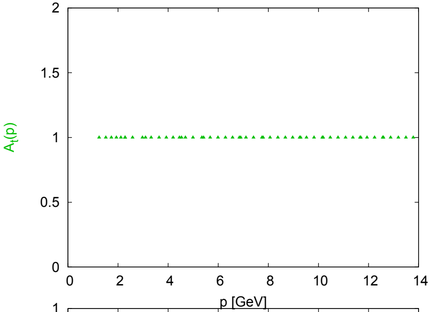

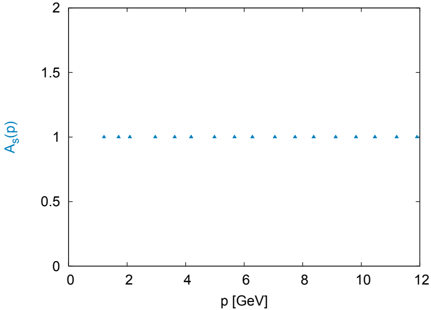

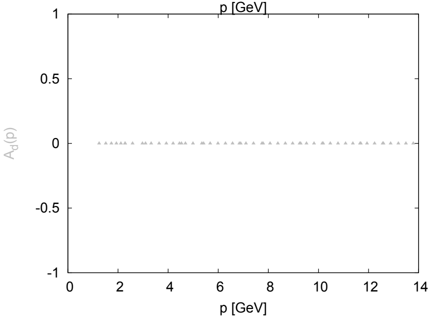

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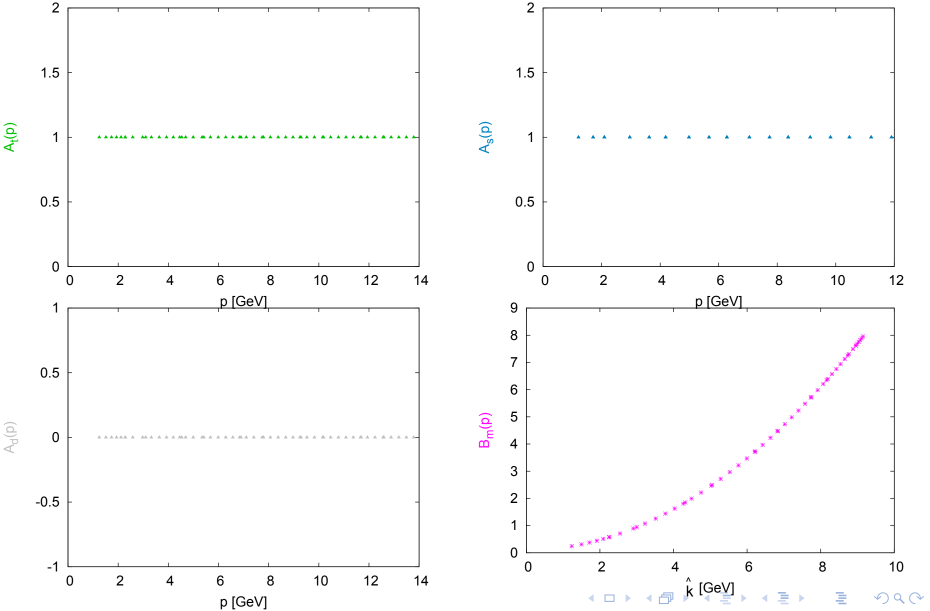
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Coulomb gauge on the lattice

The continuum Coulomb gauge condition $\partial_i A_i(\mathbf{x}, t) = 0$ is equivalent to maximizing

$$\tilde{F}_g[A](t) = \sum_{i=1}^{3} \int d^3x \; \mathrm{tr}\left[A_i^g(\mathbf{x}, t)^2\right] \quad \forall\, t.$$

with respect to gauge transformations $g(\mathbf{x}) \in SU(N_c)$.

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Lattice Coulomb gauge condition

$$F_g[U](t) = \sum_{i=1}^{3} \sum_{\mathbf{x}} \mathrm{tr}\left[U_i^g(\mathbf{x}, t) + U_i^g(\mathbf{x}, t)^\dagger\right] \xrightarrow{!} \max. \quad \forall\, t,$$

$$U_i^g(\mathbf{x}, t) = g(\mathbf{x}) U_i(\mathbf{x}, t) g^\dagger(\mathbf{x} + \hat{i}), \quad g(\mathbf{x}) \in SU(N_c).$$

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Gribov copies

The maximum of $F_g[U](t)$ is far from being unique!

Residual gauge fixing

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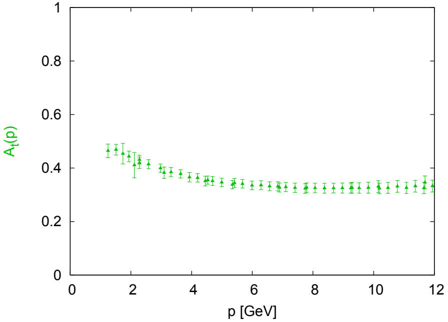
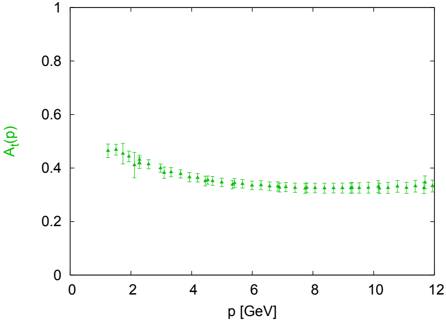
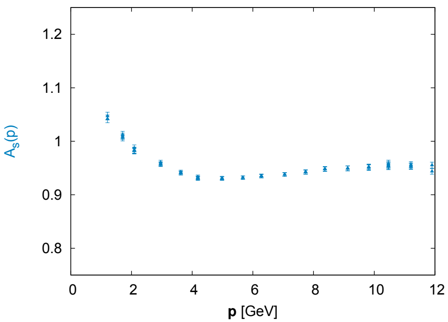
In the limit of a single spatial lattice point this procedure would gauge $U_4(\mathbf{x}, t) \to \mathbb{1}$ for all t except $t = T - 1$: *temporal gauge*.



$SU(2)$	$12^3 \times 24$
β	2.5
$a(\beta)$	$0.0855\,\mathrm{fm}$





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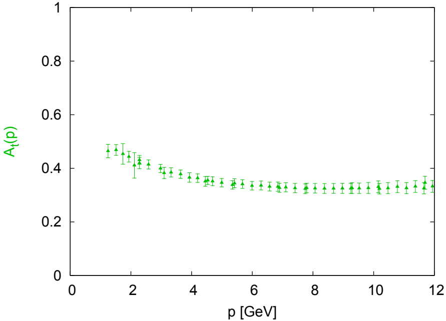

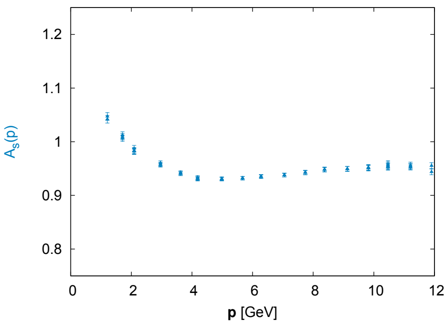

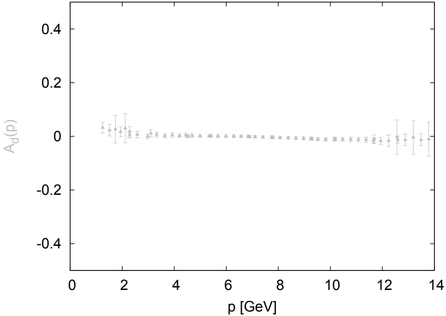

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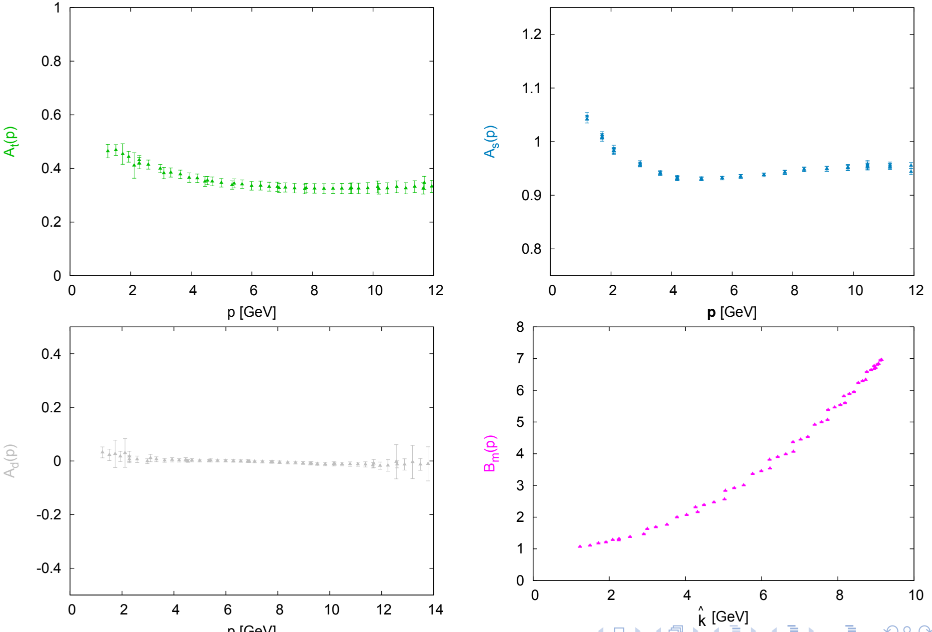
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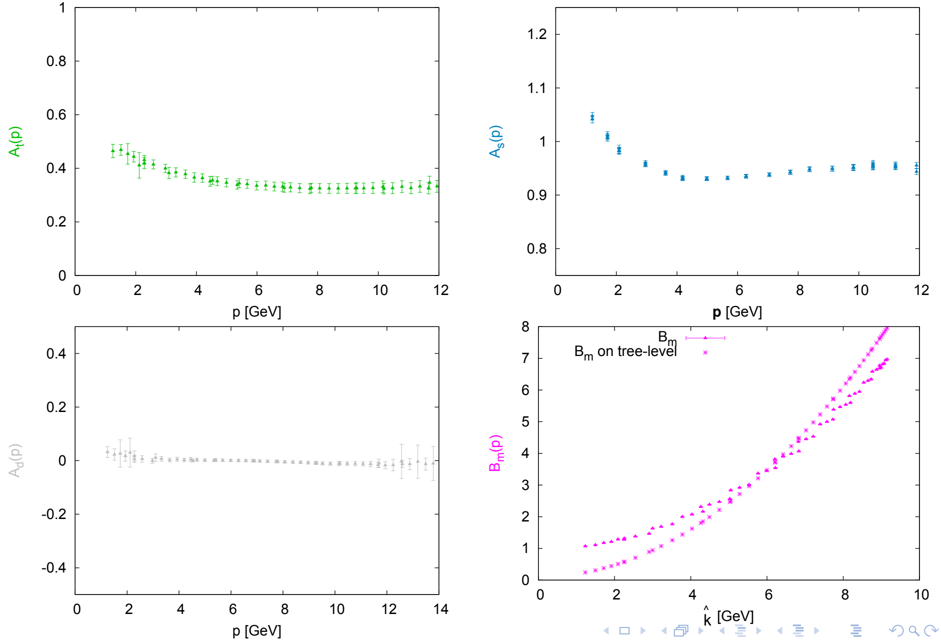
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B_m
B_m on tree-level
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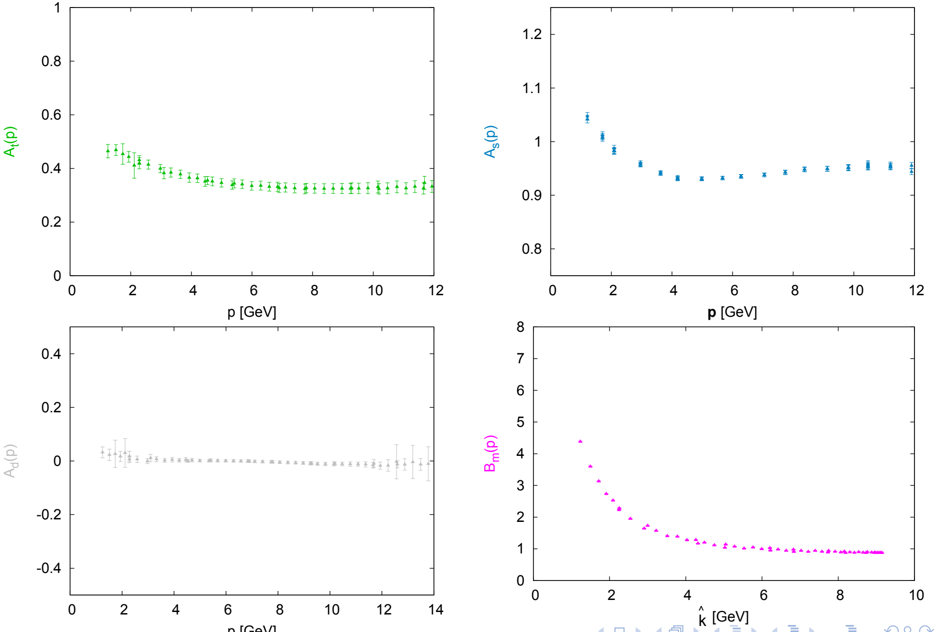
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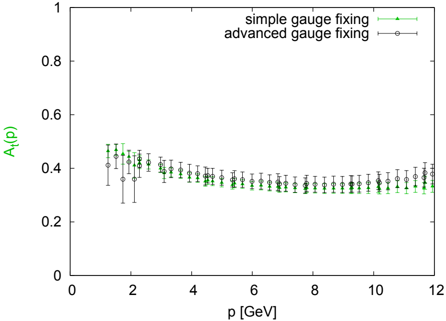

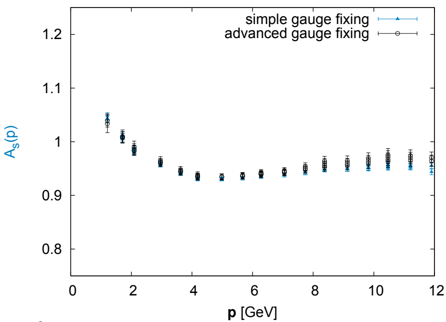

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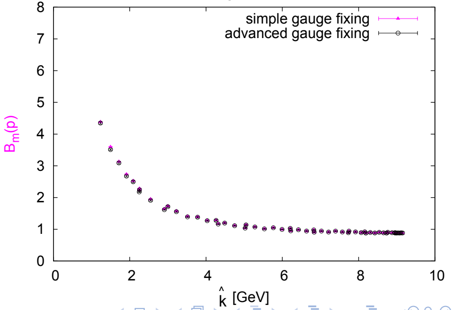

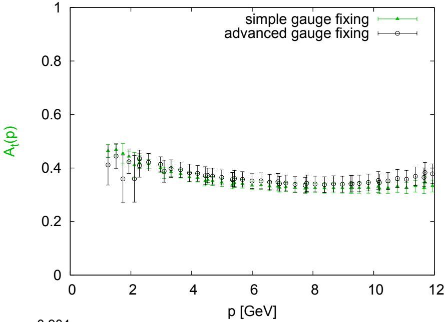
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advanced gauge fixing
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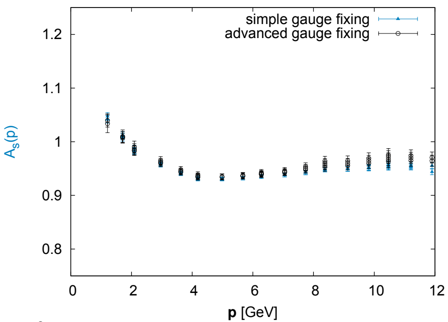
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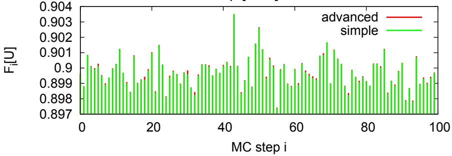
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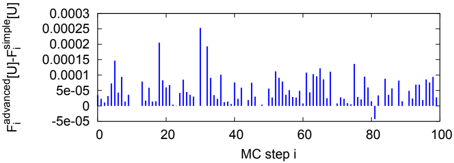
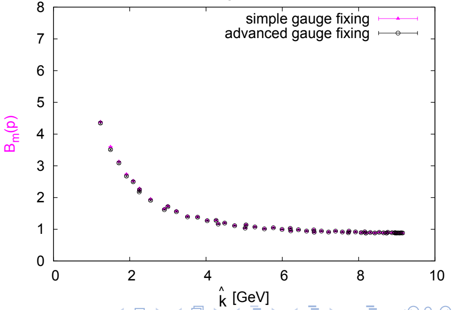
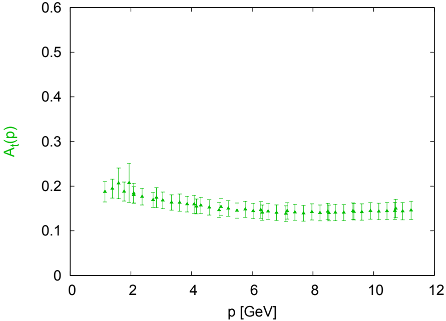
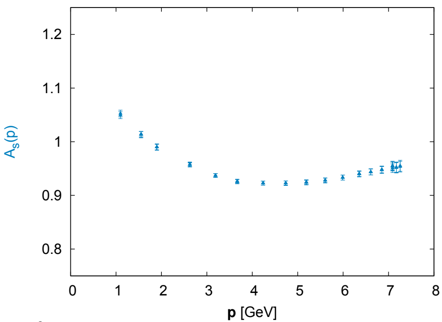
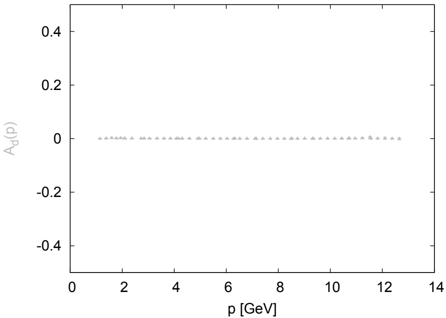
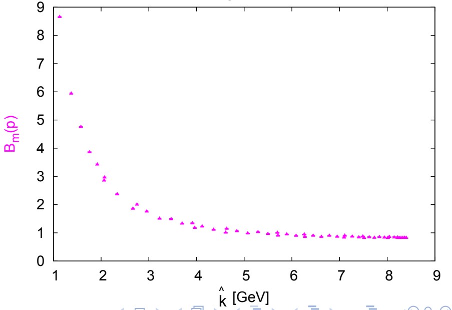

$F_i^{advanced}[U]-F_i^{simple}[U]$
MC step i

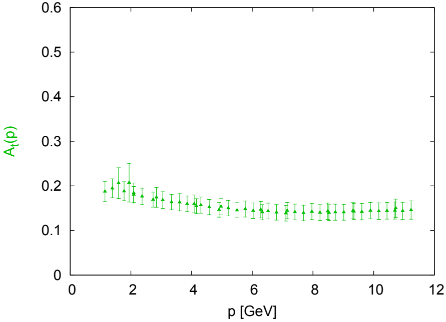

simple gauge fixing
advanced gauge fixing
$B_m(p)$
$\hat{k}$ [GeV]

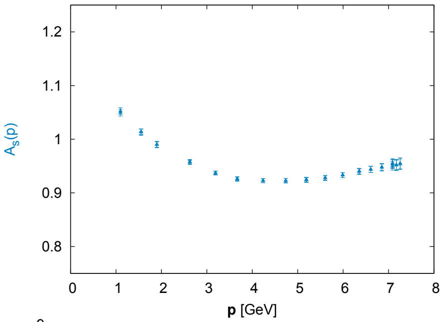

$A_t(p)$
p [GeV]


$A_s(p)$
p [GeV]

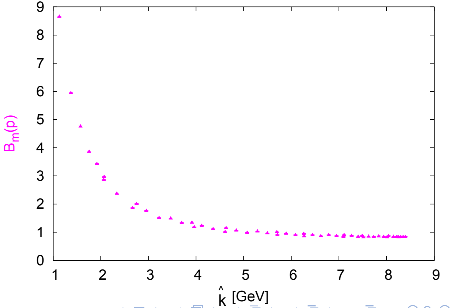

$A_d(p)$
p [GeV]

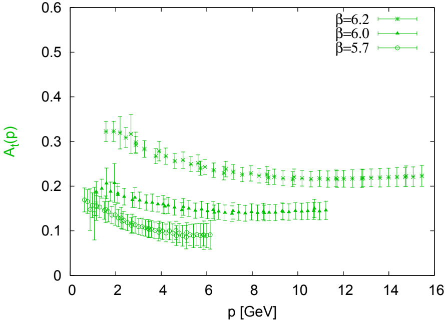

$B_m(p)$
$\hat{k}$ [GeV]

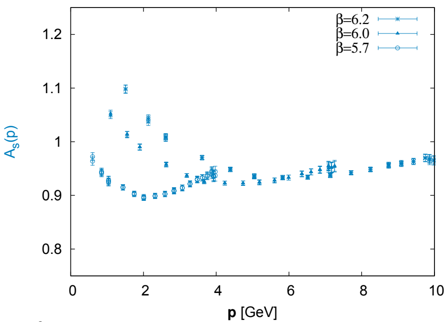




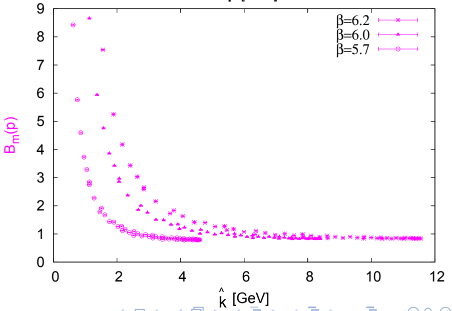
$SU(3)$	$12^3 \times 24$
β	6.0
$a(\beta)$	$0.09315\,\mathrm{fm}$
m_{bare}	$100.57\,\mathrm{MeV}$







$SU(3)$	$12^3 \times 24$	$12^3 \times 24$	$12^3 \times 24$
β	6.2	6.0	5.7
$a(\beta)$	0.0677 fm	0.09315 fm	0.170 fm
m_{bare}	100.45 MeV	100.57 MeV	101.45 MeV



Future work

- Determine dependence of dressing functions on momentum components (different residual gauge fixing schemes might help)
- Extract static propagator and check its renormalizability
- Relativistic dispersion relation
- Unquenching
- Chiral symmetric lattice action